12/8

Recall: If R is "nice" and ∂R is also nice and is a v.f. on $\mathbb{R}^3$ w/ cts partials, then

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

Ex. Calculate flux of $\vec{F} = \langle 3x, xy, 2xz \rangle$ across the surface of the cube $R = [0,1]^3$

Sol: We apply the divergence theorem

$$\begin{aligned}
\iint_{\partial R} \vec{F} \cdot d\vec{S} &= \iiint_R \operatorname{div}(\vec{F})\, dV \\
&= \iiint_R (3 + x + 2x)\, dV \\
&= \iiint_R (3+3x)\, dV \\
&= \int_0^1 \int_0^1 \int_0^1 (3+3x)\, dx\, dy\, dz \\
&= \int_0^1 \int_0^1 \left[3x + \tfrac{3}{2}x^2\right]_0^1 dy\, dz \\
&= \int_0^1 \int_0^1 \tfrac{9}{2}\, dy\, dz \\
&= \int_0^1 \left[\tfrac{9}{2}y\right]_0^1 dz \\
&= \int_0^1 \tfrac{9}{2}\, dz = \left[\tfrac{9}{2}z\right]_0^1 = \tfrac{9}{2} \quad \boxed{/}
\end{aligned}$$

Ex. Calculate flux of $\vec{F} = \langle x^2yz, xy^2z, xyz^2 \rangle$ across the boundary of the rectangular box $R = [0,a] \times [0,b] \times [0,c]$ for constants $a, b, c > 0$

Sol: We apply the divergence theorem

$$\begin{aligned}
\iint_{\partial R} \vec{F} \cdot d\vec{S} &= \iiint_R \operatorname{div}(\vec{F})\, dV \\
&= \iiint_R (2xyz + 2xyz + 2xyz)\, dV \\
&= 6 \iiint_R xyz\, dV \\
&= 6 \int_0^c \int_0^b \int_0^a xyz\, dx\, dy\, dz \\
&= 6 \left(\int_0^c z\, dz\right)\left(\int_0^b y\, dy\right)\left(\int_0^a x\, dx\right) \\
&= \tfrac{6}{8}\left[z^2\right]_0^c \left[y^2\right]_0^b \left[x^2\right]_0^a = \tfrac{6}{8}(a^2-0)(b^2-0)(c^2-0) = \tfrac{3}{4}a^2b^2c^2 \quad \boxed{/}
\end{aligned}$$

Ex. Calculate $\iint_S \vec{F}\cdot d\vec{S}$ of $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin xy\rangle$ across the surface bounding region w/ $z = 1-x^2$, $z=0$, $y=0$, $y+z=2$

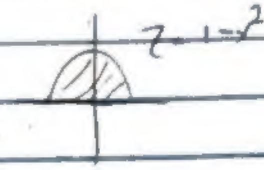
Sol: Again use divergence theorem

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, d\vec{S}$

$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[y^2+e^{xz^2}] + \frac{\partial}{\partial z}[\sin xy]$

$= y + 2y + 0 = 3y$

Parameterize R: shadow in xz-plane



$R = \{(x,y,z): -1\le x\le 1,\ 0\le z\le 1-x^2,\ 0\le y\le 2-z\}$

$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iiint_R 3y\,dV = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\,dy\,dz\,dx$

$= \int_{-1}^{1}\int_0^{1-x^2}\left[\frac{3}{2} y^2\right]_0^{2-z} dz\,dx$

$= \frac{3}{2} \int_{-1}^{1}\int_0^{1-x^2}(2-z)^2\, dz\,dx$

$= -\frac{1}{2} \int_{-1}^{1} \left[(2-z)^3\right]_0^{1-x^2} dx$

$= -\frac{1}{2}\int_{-1}^{1} \left[(2-(1-x^2))^3 - (2-0)^3\right] dx$

$= -\frac{1}{2} \int_{-1}^{1} \left[(1+x^2)^3 - 8\right]dx$

$= -\frac{1}{2}\int_{-1}^{1} (1+3x^2+3x^4+x^6-8)\,dx$

$= -\frac{1}{2}\left[x^3+\frac{3}{5}x^5+\frac{1}{7}x^7-7x\right]_{-1}^{1}$

$= -\frac{2}{2}\left(1+\frac{3}{5}+\frac{1}{7}-7\right)$ ☑

$1-x^2=0$
iff $x=\pm 1$

Ex. Compute flux of $\vec{F}=\langle xye^z, xy^2z^3, -ye^z\rangle$ across S surface of box bdd by coordinate planes and $x=3$, $y=2$, $z=1$

Sol: Apply divergence theorem, noting $S=\partial R$ for $R=[0,3]\times[0,2]\times[0,1]$ and $\operatorname{div}(\vec{F}) = ye^z + 2xyz^3 - ye^z = 2xyz^3$

$\therefore \iint_S \vec{F}\cdot d\vec{S} = \iiint_R 2xyz^3 = 2\left(\int_0^3 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^1 z^3\,dz\right)$

$= 2\left[\frac{1}{2}x^2\right]_0^3\left[\frac{1}{2} y^2\right]_0^2\left[\frac{1}{4} z^4\right]_0^1$

$= \frac{1}{8}(3^2-0)(2^2-0)(1^2-0) = \frac{9}{2}$ ☑

Ex. Compute flux of $\vec{F}=\langle z, y, zx\rangle$ across surface enclosed by coordinate planes and plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ for constant $a,b,c>0$

$\vec{n}\cdot(\vec{x}-\vec{p})=0$

$\vec{n}\cdot\vec{x} = d$

$\langle \frac{1}{a},\frac{1}{b},\frac{1}{c}\rangle\cdot\langle x,y,z\rangle = 1$

Parameterize the tetrahedron:

$R = \{(x,y,z) \mid 0 \le x \le a,\ 0 \le y \le b(1 - \frac{x}{a}),$
$0 \le z \le c(1 - \frac{x}{a} - \frac{y}{b})\}$

$\frac{x}{a} + \frac{y}{b} = 1$

and $\text{div}(\vec{F}) = 0 + 1 + x = 1 + x$

$\therefore$ by divergence theorem:

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \text{div}(\vec{F})\, dV$

$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx$

$= \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} [z]_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} = \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} c(1 - \frac{x}{a} - \frac{y}{b})\, dy\, dx$

$= c\int_0^a (1+x) \left[y - \frac{xy}{a} - \frac{1}{2b}y^2\right]_0^{b(1-\frac{x}{a})} dx$

$= c\int_0^a (1+x) \left[b(1-\frac{x}{a})(1 - \frac{x}{a} - \frac{1}{2b}b(1-\frac{x}{a}))\right] dx$

$= bc\int_0^a (1+x)(1-\frac{x}{a})((1-\frac{x}{a}) - \frac{1}{2}(1-\frac{x}{a}))\, dx$

$= \frac{1}{2}bc\int_0^a (1+x)(1-\frac{x}{a})^2\, dx$

$= \frac{1}{2}bc\int_0^a (1 + (1 - \frac{1}{a} - \frac{1}{a})x + (-\frac{1}{a} - \frac{1}{a} + \frac{1}{a^2})x^2 + \frac{1}{a^2}x^3\, dx$

$= \frac{1}{2}bc\int_0^a 1 + (1 - \frac{2}{a})x + (\frac{1}{a^2} - \frac{2}{a})x^2 + \frac{1}{a^2}x^3)\, dx$

$= \frac{1}{2}bc\left[x + \frac{1}{2}(1-\frac{2}{a})x^2 + \frac{1}{3}(\frac{1}{a^2} - \frac{2}{a})x^3 + \frac{1}{4a^2}x^4\right]_0^a$

$= \frac{1}{2}bc\left[a + \frac{a^2}{2}(1-\frac{2}{a}) + \frac{a^3}{3}(\frac{1}{a^2} - \frac{2}{a}) + (\frac{1}{4}a^2) - 0\right]$

$= \frac{1}{2}abc\,(1 + \frac{1}{2}(a-2) + \frac{1}{3}(1-2a) + \frac{1}{4}a)$

$= \frac{1}{2}abc\,(\frac{1}{3} + a(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}))$

Ex. Compute the flux of $\vec{F} = \langle 2x^3 + y^2, y^2 + z^2, 3y^2 z\rangle$ across the surface of the region bdd by paraboloid $z = 1 - x^2 - y^2$ and plane $z = -3$

Sol: Apply the divergence theorem

$\text{div}(\vec{F}) = 6x^2 + 3y^2 + 3y^2$
$= 6(x^2 + y^2)$

Parameterize R: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$-3 = 1 - x^2 - y^2$
$x^2 + y^2 = 4$

$R_{cyl} = \{(r,\theta,z) \mid 0 \le r \le 2$
$0 \le \theta \le 2\pi$
$-3 \le z \le 1 - r^2$

$\therefore \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \text{div}(\vec{F})\, dV$
$= \iiint_{R_{cyl}} \text{div}(\vec{F})(cyl)\, r\, dV_{cyl}$

$= \int_0^{2\pi}\int_0^{2}\int^{1-r^2} 6r^2 \cdot r\,dz\,dr\,d\theta$

$= 2\pi\int_0^2 6r^3\left[z\right]_{-3}^{1-r^2} dr$

$= 12\pi\int_0^2 r^3\left[1-r^2-(-3)\right] dr$ $\quad r^2 = 4-u \quad u(2)=0, \; u(0)=4$

$= 12\pi\int_0^2 r \cdot r^2(4-r^2)\,dr$ $\quad u = 4 - r^2, \; du = -2r\,dr$

$= -6\pi\int_4^0 (4-u)u\,du = 6\pi\int_0^4 4u-u^2\,du$

$= 6\pi\left(2u^2 - \frac{1}{3}u^3\right)\Big|_0^4$

$= 6\pi\left(32 - \frac{64}{3}\right)$ $\square$